Point Estimation

Chapter 10, Miller and Miller

November 24, 2015

Big Concepts of the Day

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- Key Ingredient: Random Sample
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- Today's Grand Question: Which sample statistics tell us about which population parameters?

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Examples: $\bar{X}$ is a point estimator of μ. S^2 is a point estimator of σ^2

Desirable properties of Estimators

- Unbiasedness
- Minimum Variance or Efficiency
- Consistency

Unbiased Estimator

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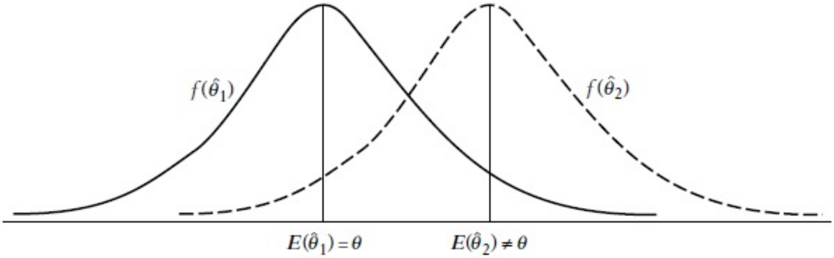
Last class we showed $E(\bar{X}) = \mu$, so $\bar{X}$ is an unbiased estimator of θ.

Unbiasedness - Intuition



Biased and unbiased estimators.

Example 1

If X is a binomial distribution with the parameters n and θ, show that the sample proportion, X/n is an unbiased estimator of θ.

Example 2

Given a random sample of size n from a population that has known mean μ and finite variance σ^2, show that

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2$$

is an unbiased estimator of σ^2.

Example 3

If X is a binomial distribution with the parameters n and θ. Is $(X+1)/(n+2)$ an unbiased estimator of θ?

Asymptotically Unbiased Estimator

Definition

Let $b_n(\theta) = E(\hat{\Theta}) - \theta$ express the bias of an estimator $\hat{\Theta}$ based on a random sample of size n from a given distribution. We say $\hat{\Theta}$ is an asymptotically unbiased estimator of θ if and only if

$$\lim_{n \to \infty} b_n(\theta) = 0$$

Example 3 - Again

If X is a binomial distribution with the parameters n and θ. Is $(X + 1)/(n + 2)$ an asymptotically unbiased estimator of θ?

Mystery of Sample Variance

If S^2 is the variance of a random sample from an infinite population with the finite variance σ^2 then $E(S^2) = \sigma^2$.

Mystery of Sample Variance

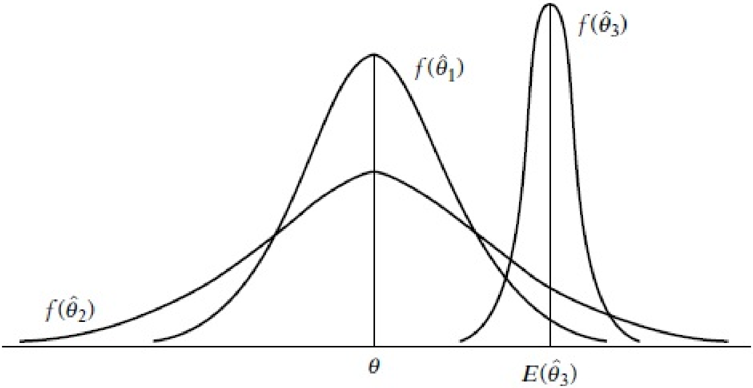
If S^2 is the variance of a random sample from an infinite population with the finite variance σ^2 then $E(S^2) = \sigma^2$. Proof.

Efficiency

Definition

If the estimate $\hat{\Theta}$ for the parameter θ of a given distribution that has the smallest variance of all unbiased estimators for θ is called the minimum variance unbiased estimator or the best unbiased estimator for θ.

Efficiency - Intuition



Distribution of three estimators of θ.

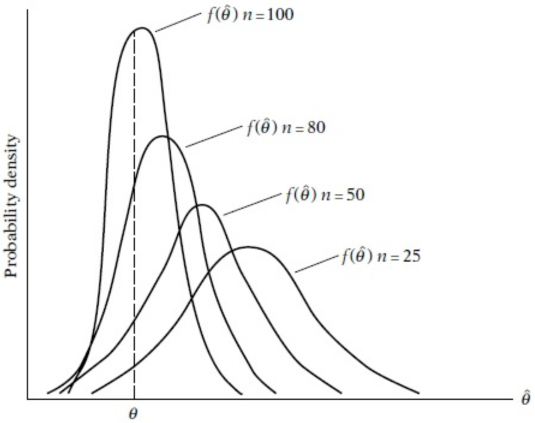
Consistency

Definition

$\hat{\Theta}$ is said to be a consistent estimator of θ if it approaches the true value θ as the sample size gets larger and larger. Formally

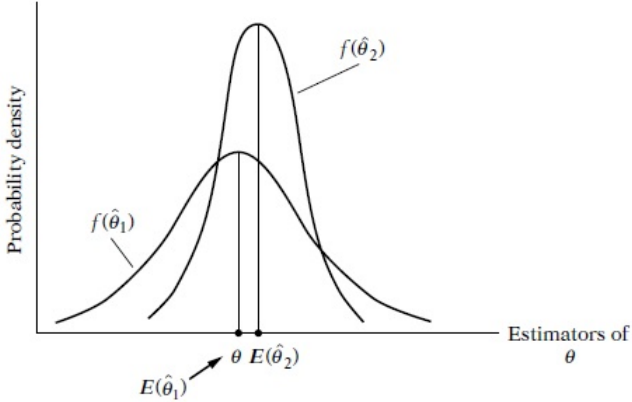
$$\lim_{n \to \infty} P(|\hat{\Theta} - \theta| < \delta) = 1 \qquad \forall \quad \delta > 0$$

Consistency - Intuition



The distribution of $\hat{\theta}$ as sample size increases.

Possible Complications



Tradeoff between bias and variance.

Ways to get (desirable) Estimators

- Method of Moments
- Method of Maximum Likelihood
- Bayesian estimation

Method of Moments

- The kth sample moment of a set of observations x_1, x_2, ... , x_n is the mean of their kth powers, and is denoted by $m_k^{'}$

$$m_k^{'} = \frac{\sum_{i=1}^{n} x_i^k}{n}$$

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- If a population has r parameters then the method of moments consists of solving the system of equations

$$m_k^{'} = \mu_k^{'} \quad k = 1, 2, \ldots, r$$

Example 4

Given a random sample of size n from a Poisson population, use the method of moments to obtain an estimator for the parameter λ.

Method of Maximum Likelihood

If x_1, x_2, ... , x_n are values of a random sample from a population with a parameter θ, the likelihood function is given by

$$L(\theta) = P(X_1 = x_1, X_2 = x_2, ..., X_n = x_n) = f(x_1, x_2, ..., x_n; \theta)$$

for values of θ within a given domain. The value of θ which maximized $L(\theta)$ is called the maximum likelihood estimator of θ.

Example 4 - Again

Given a random sample of size n from a Poisson population, use the method of maximum likelihood to obtain an estimator for the parameter λ.

Bayesian Estimation

The difference between frequentist statisticians and Bayesian statisticians has to do with whether a statistician thinks of a parameter as some unknown constant or as a random variable. Let's take a look at a simple example in an attempt to emphasize the difference.

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- Priors and Posteriors!