Functions of Random Variables

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Chapter 6 Last Half, Ross

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- $$F(Y) = P(Y \leq y) = P[u(X_1, X_2, ..., X_n) \leq y],$$
- For continuous rvs,

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Example 1

If the probability distribution of X is given by

x	0	1	2	3	4
$f(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

What is the distribution function of $Z = (X-2)^2$?

Example 1: Hint

$$x = 0, 4 \to z = 4$$
$$x = 1, 3 \to z = 1$$
$$x = 2 \to z = 0$$

So,

$$h(z = 0) = f(2)$$
$$h(z = 1) = f(1) + f(3)$$
$$h(z = 4) = f(0) + f(4)$$

Example 2

If the probability density of X is given by

$$f(x) = \begin{cases} 2xe^{-x^2} & \text{for} \quad x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

What is the distribution function of $Y = X^2$?

Example 2: Solution

$$
\begin{aligned}
F(y) &= P(Y \leq y) \\
&= P(X^2 \leq y) \\
&= P(-\sqrt{y} \leq X \leq \sqrt{y}) \\
&= \int_{x=-\sqrt{y}}^{0} 0 dx + \int_{x=0}^{\sqrt{y}} 2xe^{-x^2} dx \quad \text{[Integration mistake in class corrected!]} \\
&= \left. \frac{e^{-x^2}}{-1} \right|_{x=0}^{\sqrt{y}} \\
&= 1 - e^{-y}
\end{aligned}
$$

Note,

$$f(y) = F'(y) = e^{-y}$$

Transformation Technique: Discrete

Given X is a discrete random variable and $Y = u(X)$ where the relationship between X and Y is one-to-one, then substitute the values of Y for values of X and the probability distribution of Y is 'unchanged'.

Example 3

If the probability distribution of X is given by

x	0	1	2	3	4
$f(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

What is the distribution function of $Y = \frac{1}{1+X}$?

Transformation Technique: Continuous

Theorem: Let $f(x)$ be the value of probability density of the continuous random variable X at x. If function given by $y = u(x)$ is differentiable and either increasing or decreasing for all values within the range of X for which $f(x) \neq 0$, them for these values of x, the equation $y = u(x)$ can be uniquely solved for x to give $x = w(y)$, and for the corresponding values of y the probability density function of $Y = u(X)$ is given by

$$g(y) = f[w(y)] \cdot |w'(y)| \qquad \text{provided } u'(x) \neq 0$$

Example 4

If the probability density of X is given by

$$f(x) = \begin{cases} e^{-x} & \text{for} \quad x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

What is the distribution function of $Y = \sqrt{X}$?

Example 4: Solution

The probability density of Y is

$$g(y) = \begin{cases} 2ye^{-y^2} & \text{for} \quad y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Distribution function of Y is

$$G(y) = \begin{cases} \int_0^y 2ze^{-z^2} dz & \text{for} \quad y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

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- Write X_1 as a function of Y and X_2. Add over all values of X_2.

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- Suppose X_1 and X_2 are two discrete rvs with pdf $f(X_1, X_2)$
- What is the probability density function of $Y = u(X_1, X_2)$?
- Write X_1 as a function of Y and X_2. Add over all values of X_2.
- If X_1 and X_2 are independent rvs having Poisson distributions with parameters λ_1, and λ_2, what is the pdf of $Y = X_1 + X_2$?

Example 5

If the joint pdf of X and Y is given by

$$f(x, y) = \frac{(x-y)^2}{7}$$

for x = 1, 2 and y = 1, 2, 3, find the joint distribution of U = X+Y and V = X-Y.

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We can use this method of substitution only for linear functions of X and Y.

Theorem: MGF

If X_1, X_2, ..., X_n are independent random variables and

$$Y = X_1 + X_2 + ... + X_n$$

then

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t)$$

where $M_{X_i}(t)$ is the moment-generating function of X_i.