Joint Random Variables

Chapters 6 and parts of 7, Sheldon Ross
Chapters 4, 5, 6 (second halves), Miller & Miller

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Why multivariate?

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- Think of any other examples?

Joint probability mass function

- If X and Y are discrete random variables, the function given by

$$f(x, y) = P(X = x, Y = y)$$

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Properties

A bivariate function can serve as a joint probability density function of a pair of discrete random variables X and Y if and only if its values f(x, y) satisfy the conditions:

- $f(x, y) \geq 0$ for each pair of values (x, y) within its domain
- $\sum_x \sum_y f(x, y) = 1$ when the double summation extends over all possible pairs (x, y) within its domain.

Homework Problem

What is the joint probability mass function of X and Y when in a roll of two fair dice?
X is the largest value obtained by any dice and Y is the sum of the values.

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$$
\begin{aligned}
P_X(a) &= P(X \leq a) \\
&= P[X \leq a, Y \leq \infty] \\
&= P\left[\lim_{b \to \infty} (X \leq a, Y \leq b)\right] \\
&= \lim_{b \to \infty} P\left[(X \leq a, Y \leq b)\right] \\
&= \lim_{b \to \infty} F(a, b) \equiv F(a, \infty)
\end{aligned}
$$

How to get distribution of Y from the joint distribution of X and Y?

$$P_Y(b) = P(Y \leq b)$$
$$= \lim_{a \to \infty} F(a, b) \equiv F(\infty, b)$$

Marginal Distribution

If X and Y are discrete random variables and $f(x, y)$ is the value of their joint probability distribution at (x, y), the function given by

$$g(x) = \sum_{y} f(x, y)$$

for each $x \in X$ is called the marginal distribution of X. Correspondingly, the function given by

$$h(y) = \sum_{x} f(x, y)$$

for each $y \in Y$ is called the marginal distribution of Y.

Expected Joint Functions

If X and Y are discrete random variables and $f(x, y)$ is the joint pdf at (x, y), the expected value of $g(X, Y)$ is

$$E[g(X, Y)] = \sum_x \sum_y g(x, y) \cdot f(x, y)$$

Expectation of Sum of Joint Functions

If c_1, c_2, ... and c_n are constants, then

$$E\left[\sum_{i=1}^{n} c_i g_i(X_1, X_2, ..., X_k)\right] = \sum_{i=1}^{n} c_i E\left[g_i(X_1, X_2, ..., X_k)\right]$$

Product Moments about the Origin

The rth and sth moment about the origin of the discrete random variables X and Y, denoted by $\mu'_{r,s}$ is the expected value of $X^r Y^s$:

$$\begin{aligned} \mu'_{r,s} &= E(X^r Y^s) \\ &= \sum_x \sum_y x^r y^s \cdot f(x, y) \end{aligned}$$

Product Moments about the Mean

The rth and sth moment about the means of the discrete random variables X and Y, denoted by $\mu_{r,s}$ is the expected value of $(X-\mu_X)^r(Y-\mu_Y)^s$:

$$\begin{aligned}\mu_{r,s} &= E[(X-\mu_X)^r(Y-\mu_Y)^s] \\ &= \sum_x \sum_y (x-\mu_X)^r(y-\mu_Y)^s \cdot f(x,y)\end{aligned}$$

Product Moments about the Mean

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$\mu_{1,1}$ is covariance of X and Y or σ_{XY}.

Theorem: Covariance

$$\sigma_{XY} = \mu'_{1,1} - \mu_X \mu_Y$$

Theorem: Correlation

$$\rho_{X,Y} = \mathrm{corr}(X, Y) = \frac{\mathrm{cov}(X, Y)}{\sigma_X \sigma_Y}$$
$$= \frac{E[(X - \mu_X)(Y - \mu_Y)]}{\sigma_X \sigma_Y}$$
$$= \frac{\sum_x \sum_y (x - \mu_x)(y - \mu_y) f(x, y)}{\sqrt{\left[\sum_x (x - \mu_x)^2 f_x(x)\right] \left[\sum_y (y - \mu_y)^2 f_y(y)\right]}}$$

Independence of Random Variables

The random variables X and Y are said to be independent if, for any two sets of real numbers A and B,

$$P(X \in A, Y \in B) = P[X \in A] \cdot P[Y \in B]$$

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Alternatively

$$f(x_1, x_2, ...x_n) = f_1(x_1) \cdot f_2(x_3) \cdot \cdot f_n(x_n)$$

Homework Problem

The joint probability mass function of X and Y are given by

$$p(1,1) = \frac{1}{8}, \qquad p(1,2) = \frac{1}{4}$$
$$p(2,1) = \frac{1}{8}, \qquad p(2,2) = \frac{1}{2}$$

Are X and Y independent?

Homework Problem

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Are X and Y independent? Compute $P[X|Y \leq 1]$

Theorem: Independence

If X and Y are independent, then

$$E(XY) = E(X) \cdot E(Y) \quad \text{and} \quad \sigma_{XY} = 0$$

Theorem: More Independence

If X_1, X_2, ... , and X_n are independent, then

$$E(X_1 X_2 ... X_n) = E(X_1) \cdot E(X_2) ... E(X_n)$$

Theorem: Moments of Linear Combinations of Random Variables

If X_1, X_2, ... , and X_n are random variables and a_1, a_2, ... , and a_n are constants and

$$Y = \sum_{i=1}^{n} a_i X_i$$

then

$$E(Y) = \sum_{i=1}^{n} a_i E(X_i)$$

$$var(Y) = \sum_{i=1}^{n} a_i^2 \cdot var(X_i) + 2 \sum \sum_{i<j} a_i a_j \cdot cov(X_i, X_j)$$

Corollary of the Previous Theorem

If random variables X_1, X_2, ... , and X_n are *independent* and a_1, a_2, ... , and a_n are constants and

$$Y = \sum_{i=1}^{n} a_i X_i$$

then

$$var(Y) = \sum_{i=1}^{n} a_i^2 \cdot var(X_i)$$

Theorem: MB

If X_1, X_2, ... , and X_n are random variables and a_1, a_2, ... ,a_n, b_1, b_2, ... , b_n are constants and

$$Y_1 = \sum_{i=1}^{n} a_i X_i, \qquad Y_2 = \sum_{i=1}^{n} b_i X_i$$

then

$$cov(Y_1, Y_2) = \sum_{i=1}^{n} a_i b_i \cdot var(X_i) + \sum\sum_{i<j} (a_i b_j + a_j b_i) \cdot cov(X_i, X_j)$$

Corollary of the Previous Theorem

If random variables X_1, X_2, ... , and X_n are independent and a_1, a_2, ... ,a_n, b_1, b_2, ... , b_n are constants and

$$Y_1 = \sum_{i=1}^{n} a_i X_i, \qquad Y_2 = \sum_{i=1}^{n} b_i X_i$$

then

$$cov(Y_1, Y_2) = \sum_{i=1}^{n} a_i b_i \cdot var(X_i)$$

Conditional Distribution

If $f(x, y)$ is the value of their joint probability distribution of discrete random variables X and Y at (x, y), and $h(y)$ is the value of the marginal distribution of Y at y, the function given by

$$v(x|y) = \frac{f(x, y)}{h(y)} \qquad h(y) \neq 0$$

for each $x \in X$ is called the conditional distribution of X given Y =y. Correspondingly, if $g(x)$ is the value of the marginal distribution of X at x, the function given by

$$w(x|y) = \frac{f(x, y)}{g(x)} \qquad g(x) \neq 0$$

for each $y \in Y$ is called the conditional distribution of Y given X = x.

Homework Problem

If

$$var(X_1) = 5, var(X_2) = 4, var(X_3) = 7,$$
$$cov(X_1, X_2) = 3, cov(X_1, X_3) = -2,$$

and X_2 and X_3 are independent. Find the covariance of $Y_1 = X_1 - 2X_2 + 3X_3$ and $Y_2 = -2X_1 + 3X_2 + 4X_3$.