Continuous Random Variables

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8th October 2015

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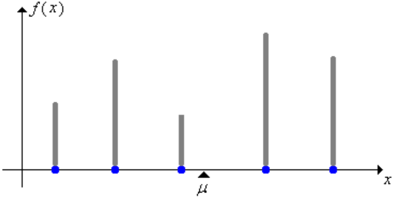
Continuous RVs

- While a discrete random variable takes either finite or countably infinite possible values, a continuous random variable takes uncountable possible values
- A function $f(x)$ with values defined over a set of real numbers is called a probability density function of the continuous variable X if and only if

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

for any real constants a and b with $a \leq b$.

Probability Functions



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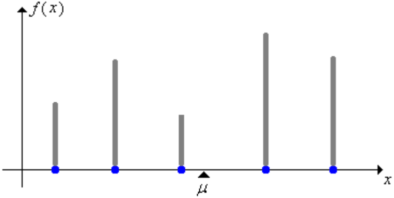

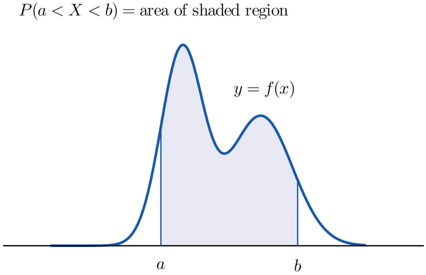

Theorem: Courtesy Continuity

If X is a continuous random variable and a and b are real constants with $a \leq b$, then

$$\begin{aligned} P(a \leq X \leq b) &= P(a \leq X < b) \\ &= P(a < X \leq b) \\ &= P(a < X < b) \end{aligned}$$

Theorem: Probability Density

A function can be considered as the probability density of a continuous random variable X if and only if its values, $f(x)$, satisfy the conditions

1. $f(x) \geq 0$ for $-\infty < x < \infty$
2. $\int\limits_{-\infty}^{\infty} f(x)dx = 1$

Example 1

What values of k can

$$f(z) = \begin{cases} kz\exp^{-z^2} & \text{for} \quad z > 0 \\ 0 & \text{for} \quad z \leq 0 \end{cases}$$

be a probability density function of a random variable?

Distribution Function

If X is a continuous random variable, and the probability density at x is $f(x)$, then the function F

$$F(a) = P(X \leq a) = \int_{-\infty}^{a} f(x)dx$$

is called the distribution function or the cumulative distribution function of X.

Theorem: pdf – cdf relation

If $f(x)$ and $F(x)$ are the values of the probability density and distribution function of X at x then

1. $P(a \leq X \leq b) = F(b) - F(a)$ for any real constants a and b with $a \leq b$
2. $f(x) = \frac{dF(x)}{dx}$ where the derivative exists.

Intuition?

Example 2

The distribution function of a random variable X is given by

$$F(x) = \begin{cases} 0 & \text{for} \quad x < -1 \\ \frac{x+1}{2} & \text{for} \quad -1 \leq x \leq 1 \\ 1 & \text{for} \quad x \geq 1 \end{cases}$$

Find $P(-0.5 < X < 3)$.

Expectation

If X is a discrete random variable and $f(x)$ is the pdf, the expected value of X is

$$E(X) = \sum_x x \cdot f(x)$$

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If X is a continuous random variable and $f(x)$ is the pdf, the expected value of X is

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

Example 3

Given

$$f(x) = \begin{cases} x & \text{for} \quad 0 < x \leq 1 \\ \frac{1}{2} & \text{for} \quad 1 < x \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

Find $E(X)$.

Theorem: Expected Functions

If X is a discrete random variable and $f(x)$ is the pdf, the expected value of $g(X)$ is

$$E[g(X)] = \sum_{x} g(x) \cdot f(x)$$

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Powerful result.

Theorem: Expectation of Simple Function

If a and b are constants and X is *any* random variable, then

$$E(aX + b) = aE(X) + b$$

Theorem: Expectation of Sum of Functions

If c_1, c_2, ... and c_n are constants, then

$$E\left[\sum_{i=1}^{n} c_i g_i(X)\right] = \sum_{i=1}^{n} c_i E\left[g_i(X)\right]$$

Example 4

Given

$$f(x) = \begin{cases} x & \text{for} & 0 < x \leq 1 \\ \frac{1}{2} & \text{for} & 1 < x \leq 2 \\ 0 & \text{elsewhere} & \end{cases}$$

Find the expected value of $X^2 - 5X + 3$.

Theorem: Moments About Origin

The rth moment about origin of a discrete random variable X, for r = 0, 1, 2, ..., $\mu_r^{'}$ is the expected value of X^r:

$$\mu_r^{'} = E[X^r] = \sum_x x^r \cdot f(x)$$

Theorem: Moments About Origin

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The rth moment about origin of a <u>continuous</u> random variable X, for r = 0, 1, 2, ..., $\mu_r^{'}$ is the expected value of X^r:

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$\mu_1^{'}$ is mean or expectation of X, $E(X)$.

Theorem: Moments About Mean

The rth moment about mean of a discrete random variable X, for r $= 0, 1, 2, ...$, μ_r is the expected value of X^r:

$$\mu_r = E[(X - \mu)^r]$$
$$= \sum_x (x - \mu)^r \cdot f(x)$$

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What is μ_2?

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What is μ_2? Variance

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What is μ_3? Measure of Skewness

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What is μ_4?

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What is higher moments about mean (central moments)?

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What is μ_2? Variance
What is μ_3? Measure of Skewness
What is μ_4? Measure of Kurtosis
What is higher moments about mean (central moments)? Yield the shape of the probability density curve.

Theorem: Variance

$$Var[X] = E[(X - \mu)^2] = E[X^2] - (E[X])^2$$

Theorem: Variance

If X has the variance σ^2, then

$$var(aX + b) = a^2\sigma^2$$

where a and b are two constants.

Moment-Generating Functions

The moment generating function of a discrete random variable X, where it exists, is given by:

$$M_X(t) = E[e^{tX}] = \sum_x e^{tx} \cdot f(x)$$

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The moment generating function of a continuous random variable X, where it exists, is given by:

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)dx$$

Theorem: Moments Generation

$$\frac{d^r M_X(t)}{dt^r}\Big|_{t=0} = \mu_r^{'}$$

Theorem: Moments-Generating Functions

If a and b are constants, then

$$M_{X+a}(t) = E\left[e^{(X+a)t}\right] = e^{at} \cdot M_X(t)$$
$$M_{bX}(t) = E\left[e^{(bX)t}\right] = M_X(bt)$$
$$M_{\frac{X+a}{b}}(t) = E\left[e^{\left(\frac{X+a}{b}\right)t}\right] = e^{\frac{at}{b}} \cdot M_X\left(\frac{t}{b}\right)$$

Uniform Random Variable

A random variable is said to be uniformly distributed over the interval (α, β) if it is "equal chances" of getting any value within this interval. The probability of X falling in a subinterval of (α, β) depends only on the length of the subinterval.

Parameters	$\alpha, \beta > 0$
PDF	$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & \text{for} \quad \alpha < x < \beta \\ 0 & \text{otherwise} \end{cases}$
CDF	$F(x) = \begin{cases} 0 & \text{for} \quad x < \alpha \\ \frac{x-\alpha}{\beta-\alpha} & \text{for} \quad \alpha < x < \beta \\ 1 & \text{for} \quad x > \beta \end{cases}$
$E(X)$	$\frac{\alpha+\beta}{2}$
$Var(X)$	$\frac{(\beta-\alpha)^2}{12}$
MGF of X	$\frac{e^{t\beta}-e^{t\alpha}}{t(\beta-\alpha)}$

Normal Random Variables

This distribution is most easily recognized, aesthetically pleasing and famous for its bell-shape. The density function is symmetric about its mean. The curve is unimodal. It describes many real world phenomenon like distribution of height, distribution of weight, some psychological test scores, etc.

Parameters	$\mu \in \mathbb{R}$, $\sigma^2 > 0$	
PDF	$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$	$-\infty < x < \infty$
CDF	Complicated	
$E(X)$	μ	
$Var(X)$	σ^2	
MGF of X	$exp\{\mu t + \frac{\sigma^2 t^2}{2}\}$	

Exponential Random Variable

Exponential distribution is often associated with the amount of time until some event occurs. For instance, starting from now until earthquake occurs, or until a war breaks down, etc.

Parameters	$\lambda > 0$
PDF	$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for} \quad x \geq 0 \\ 0 & \text{otherwise} \end{cases}$
CDF	$F(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 1 - e^{-\lambda x} & \text{for} \quad x \geq 0 \end{cases}$
$E(X)$	$\frac{1}{\lambda}$
$Var(X)$	$\frac{1}{\lambda^2}$
MGF of X	$\frac{\lambda}{\lambda - t}$