Random Variables

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- Why do we conduct experiments?
- Because we are interested in some particular outcomes
- Example, in tossing two dice we are interested in the sum of two dice, not on the value of each die
- In teacher ratings, we are interested in the average scores not in individual scores
- If S is a sample space with a probability measure and X is a real-valued function defined over the elements of S, then X is called a random variable.

Intuition of Random Variables

Suppose a medical researcher is testing eight elderly adults for their allergic reaction (yes or no) to a new drug for controlling blood pressure.

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1. Random variable (X) is the number of people who have allergic reaction (random variable denotes outcome of an experiment)
2. Possible values of X are 0, 1, 2, 3, 4, 5, 6, 7, 8 (random variable is a number)
3. If p is the probability of allergic reaction, the probability of different values of X is (each value of the random variable occurs with a probability)

$P(X=0) = (1-p)^8$	$P(X=5) = {}^8C_5 \cdot p^5 \cdot (1-p)^3$
$P(X=1) = {}^8C_1 \cdot p \cdot (1-p)^7$	$P(X=6) = {}^8C_6 \cdot p^6 \cdot (1-p)^2$
$P(X=2) = {}^8C_2 \cdot p^2 \cdot (1-p)^6$	$P(X=7) = {}^8C_7 \cdot p^7 \cdot (1-p)^1$
$P(X=3) = {}^8C_3 \cdot p^3 \cdot (1-p)^5$	$P(X=8) = p^8$
$P(X=4) = {}^8C_4 \cdot p^4 \cdot (1-p)^4$	

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- the sum of the two rolls;
- Ans: $\{2, 3, 4, ..., 12\}$
- the value of the first roll minus the value of the second roll
- Ans: $\{-5, -4, ..., 4, 5\}$

Example 2

Two balls are chosen randomly from an urn containing 8 white, 4 black and 2 orange balls. Suppose that we win Rs. 2 for each black ball selected and we lose Rs. 1 for each white ball. Let X denote our winnings.

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What are the possible values of X?
What are the probabilities associated with each value?

Definition

- A discrete random variable takes at most countable number of possible values
- It is a random variable X which assumes one the values x_1, x_2,...
- The probability mass function is

$$
\begin{aligned}
p(x_i) &\geq 0 \qquad \text{for i} = 1, 2, 3.. \\
p(x) &= 0 \qquad \text{for all other values of x}
\end{aligned}
$$

- $\sum_{i=1}^{\infty} p(x_i) = 1$

Example 3

Four independent flips of a fair coin are made. Let X denote the number of heads obtained. Plot the probability mass function of the random variable X.

Probability Distribution

If X is a discrete random variable, the function given by $f(x) = P(X = x)$ for each x within the range of X is called the probability distribution of X.

Probability Distribution

A function can be considered as the probability distribution of a discrete random variable X if and only if its values, $f(x)$, satisfy the conditions

1. $f(x) \geq 0$ for each value within its domain
2. $\sum_x f(x) = 1$, where the summation extends over all the values within its domain.

Example 4

What values of k can

$$f(x) = (1-k)k^x$$

serve as the values of the probability distribution of a random variable with the countably infinite range $x = 0, 1, 2, ...$?

Distribution Function or Cumulative Distribution

If X is a discrete random variables, the cumulative distribution function F can be expressed as

$$F(a) = \sum_{\text{all } x \leq a} p(x)$$

where p(x) is the value of probability distribution of X at x.

Example 5

Let

$$f(x) = \frac{{}^2C_x \, {}^4C_{3-x}}{{}^6C_3} \qquad \text{for } x = 0, 1, 2.$$

Find the distribution function.

Properties of Distribution Function

The values $F(x)$ of the distribution function of a discrete random variable X satisfies the following properties:

1. $F(-\infty) = 0$ and $F(\infty) = 1$
2. If $a < b$, then $F(a) \leq F(b)$ for any real number a and b

Intuition?

Theorem: pdf - cdf relation

If the range of a random variable X consists of the values $x_1 < x_2 < x_3 < \ldots < x_n$, then

1. $f(x_1) = F(x_1)$
2. $f(x_i) = F(x_i) - F(x_{i-1})$ for $i = 2, 3, \ldots, n$

Intuition?

Expectation

If X is a discrete random variable and $f(x)$ is the pdf, the expected value of X is

$$E(X) = \sum_x x \cdot f(x)$$

Theorem: Expected Functions

If X is a discrete random variable and $f(x)$ is the pdf, the expected value of $g(X)$ is

$$E[g(X)] = \sum_x g(x) \cdot f(x)$$

Theorem: Expected Functions

If X is a discrete random variable and $f(x)$ is the pdf, the expected value of $g(X)$ is

$$E[g(X)] = \sum_{x} g(x) \cdot f(x)$$

Powerful result.

Example 6

What is $E[X^2]$?

Example 6

What is $E[X^2]$? Given your probability distributions, calculate the expected value of the profit function $\pi(X) = X^2 - 5X + 3$.

Theorem: Expectation of Simple Function

If a and b are constants, then

$$E(aX + b) = aE(X) + b$$

Theorem: Expectation of Sum of Functions

If c_1, c_2, ... and c_n are constants, then

$$E\left[\sum_{i=1}^{n} c_i g_i(X)\right] = \sum_{i=1}^{n} c_i E\left[g_i(X)\right]$$

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Use this shortcut to calculate the expected profit of your firm.

Theorem: Moments About Origin

The rth moment about origin of a discrete random variable X, for r = 0, 1, 2, ..., μ_r' is the expected value of X^r:

$$\mu_r' = E[X^r] = \sum_x x^r \cdot f(x)$$

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What is μ_1'?

Theorem: Moments About Mean

The rth moment about mean of a discrete random variable X, for r = 0, 1, 2, ..., μ_r is the expected value of X^r:

$$\mu_r = E[(X - \mu)^r]$$
$$= \sum_x (x - \mu)^r \cdot f(x)$$

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What is μ_2?

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What is μ_2? What does it mean?

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What is μ_2? What does it mean?
What is μ_3?

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What is μ_4?

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$$\mu_r = E[(X - \mu)^r]$$
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What is μ_2? What does it mean?
What is μ_3? What does it mean?
What is μ_4? What does it mean?
What is higher moments about mean (central moments)?

Theorem: Variance

$$Var[X] = E[(X-\mu)^2] = E[X^2] - (E[X])^2$$

Theorem: Variance

If X has the variance σ^2, then

$$Var(aX + b) = a^2\sigma^2$$

where a and b are two constants.

Theorem: Chebyshev's

If μ and σ are the mean and the standard deviation of a random variable X, then for any positive constant k the probability is at least $1 - 1/k^2$ that X will take on value within k standard deviations of the mean. Symbolically,

$$P(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}, \qquad \sigma \neq 0$$

Theorem: Chebyshev's

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Proof not required.

Moment-Generating Functions

The moment generating function of a discrete random variable X, where it exists, is given by:

$$M_X(t) = E[e^{tX}] = \sum_x e^{tx} \cdot f(x)$$

Theorem: Moments Generation

$$\frac{d^r M_X(t)}{dt^r}\Big|_{t=0} = \mu_r^{'}$$

Theorem: Moments-Generating Functions

If a and b are constants, then

$$M_{X+a}(t) = E\left[e^{(X+a)t}\right] = e^{at} \cdot M_X(t)$$

$$M_{bX}(t) = E\left[e^{(bX)t}\right] = M_X(bt)$$

$$M_{\frac{X+a}{b}}(t) = E\left[e^{\left(\frac{X+a}{b}\right)t}\right] = e^{\frac{at}{b}} \cdot M_X\left(\frac{t}{b}\right)$$

Bernoulli Random Variable

Bernoulli random variable (X) captures success or failure of a trial. If the experiment is a success the X equals one else X equals zero.

Parameters	PDF	CDF
$0 \leq p \leq 1$	$f(0) = 1 - p$ $f(1) = p$	$F(0) = 1 - p$ $F(1) = 1$

$E(X) = p$
$Var(X) = p(1 - p)$
MGF of X $= 1 - p + pe^t$

Binomial Random Variable

Binomial random variable (X) denotes the probability of i success from n experiments. If n independent trials or games or experiments are conducted, each of which has the probability of success equal to p then f(i) refers to the probability that i of these n experiments were successful and rest were not.

Parameters	PDF	CDF
$n > 0,\ 0 \le p \le 1$	$f(X = i) = {}^nC_i p^i (1-p)^{n-i}$	$P(X \le i) = \sum_{k=0}^{i} {}^nC_k p^k (1-p)^{n-k}$

$E(X) = np$
$Var(X) = np(1-p)$
MGF of X $= (1 - p + pe^t)^n$

Poisson Random Variable

Poisson distribution is the probability distribution for a very large number of independent rare ($p \approx 0$) events. Let us say we have a hundred apple trees, all identical.

Parameters	PDF	CDF
$\lambda > 0$	$f(X = i) = e^{-\lambda} \frac{\lambda^i}{i!}$	$P(X \leq i) = \sum_{k=0}^{i} e^{-\lambda} \frac{\lambda^k}{k!}$

$E(X) = \lambda$
$Var(X) = \lambda$
MGF of X $= exp(\lambda(e^t - 1))$

Geometric Random Variable

The geometric random variable (X) describes the probability of being successful after a series of failures.

Parameters	PDF	CDF
$0 \leq p \leq 1$	$f(X = i) = p(1-p)^{i-1}$	$P(X \leq i) = \sum_{k=1}^{i} p(1-p)^{k-1}$

$E(X) = {}^{1}/_{p}$
$Var(X) = {}^{(1-p)}/_{p^2}$
MGF of X $= {}^{pe^t}/_{(p+(1-p)e^t)}$

Negative Binomial Random Variable

Negative Binomial Random Variable depicts how many trials need to be done to get r successes. For the rth success to occur in the nth trial then r-1 successes should be in the first n-1 trials and the nth trial must be a success.

Parameters	PDF	CDF
r, $0 \leq p \leq 1$	$f(X = i) = {}^{n-1}C_{r-1} p^r (1-p)^{n-r}$	$P(X \leq i) = \sum_{k=n}^{i} {}^{k-1}C_{r-1} p^r (1-p)^{k-r}$

$E(X) = {}^{r}/_{p}$
$Var(X) = {}^{r(1-p)}/_{p^2}$
MGF of X $= \left[{}^{pe^t}/_{(p+(1-p)e^t)}\right]^r$