Conditional Probability and Independence

Chapter 3, Sheldon Ross

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- So, what's the answer?

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- So, probability of 'first die is 3' given 'the sum of dice is 8' equals $1/5$.

Definition

If $P(F) > 0$, then

$$P(E|F) = \frac{P(E \cap F)}{P(F)}.$$

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- P(passed all three exams) $=$

$$\begin{aligned} P(F \cap S \cap T) &= P(T|S \cap F)P(S \cap F) \\ &= P(T|S \cap F)P(S|F)P(F) \\ &= .7 * .8 * .9 = .504 \end{aligned}$$

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\begin{aligned}
P(Fd|Sd \cap Td) &= \frac{P(Fd \cap Sd \cap Td)}{P(Sd \cap Td)} \\
P(Fd \cap Sd \cap Td) &= \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} \\
P(Sd \cap Td) &= P(Fd^c \cap Sd \cap Td) + P(Fd \cap Sd \cap Td) \\
&= \frac{39}{52} \cdot \frac{13}{51} \cdot \frac{12}{50} + \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} \\
P(Fd|Sd \cap Td) &= 11/50 = .22
\end{aligned}
$$

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$$\begin{aligned} E &= E \cap (F \cup F^c) \\ &= (E \cap F) \cup (E \cap F^c) \\ P(E) &= P(E \cap F) + P(E \cap F^c) \\ &= P(E|F)P(F) + P(E|F^c)P(F^c) \\ &= P(E|F)P(F) + P(E|F^c)[1 - P(F)] \end{aligned}$$

Bayes's Formula: Definition 2

$$P(E|F) = \frac{P(F|E)P(E)}{P(F)}$$

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P(failed second exam|did not pass all exams) ?

P(failed second exam|did not pass all exams) = ?
P(did not pass all exam)

$$= P(F^c) + P(F \cap S^c) + P(F \cap S \cap T^c)$$
$$= [1 - P(F)] + P(S^c|F)P(F) + P(T^c|F \cap S)P(F \cap S)$$
$$= [1 - P(F)] + [1 - P(S|F)]P(F) + [1 - P(T|F \cap S)]P(S|F)P(F)$$
$$= .1 + .18 + .216 = .496$$

Ans $= .18/.496 = 0.3629$

Odds of an Event

The odds of an event A is defined by

$$\frac{P(A)}{P(A^c)} = \frac{P(A)}{1 - P(A)}$$

How do odds change in presence of new information?

$$\frac{P(A|E)}{P(A^c|E)} = \frac{P(A)P(E|A)}{P(A^c)P(E|A^c)}$$

Example 5

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What if the population consists of twice as many males as females?

Example 6

A sample of 3 is drawn in this manner:

- Start with an urn with 5 white and 7 red balls
- At each stage ball drawn and color noted.
- Ball returned to urn along with another ball of same color

What is the probability that the sample will contain 2 white balls?

Example 6: Solution

2 white balls are possible: RWW, WRW, WWR

- $P(RWW) = \frac{7}{12}\frac{5}{13}\frac{6}{14}$
- $P(WRW) = \frac{5}{12}\frac{7}{13}\frac{6}{14}$
- $P(WWR) = \frac{5}{12}\frac{6}{13}\frac{7}{14}$

Ans: 15/52

Independent Events

Two events E and F are said to be independent if $P(E \cap F) = P(E)P(F)$.

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- E = event that it will rain tomorrow
  F = event that it will rain day after tomorrow

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Proof:

$$E = E \cap S = E \cap (F \cup F^c) = (E \cap F) \cup (E \cap F^c)$$

As $(E \cap F)$ and $(E \cap F^c)$ are mutually exclusive,

$$\begin{aligned} P(E) &= P(E \cap F) + P(E \cap F^c) \\ P(E) &= P(E)P(F) + P(E \cap F^c) \\ P(E \cap F^c) &= P(E) - P(E)P(F) \\ &= P(E)[1 - P(F)] = P(E)P(F^c) \end{aligned}$$

Example 8

Independent trials of rolling a pair of dice are performed. What is the probability that an outcome of 5 appears before an outcome of 7 when the outcome is sum of roll of dice?

Example 8: Solution

E_n = event when neither 5 or 7 appeared in $n-1$ trials and 5 appeared in n^{th} trial. We want

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n)$$

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$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n) = \frac{1}{9}\sum_{n=1}^{\infty}\left(\frac{13}{18}\right)^{n-1} = \frac{2}{5}$$

Is $P(\cdot|F)$ a probability?

- $0 \leq P(E|F) \leq 1$
- $P(S|F) = 1$
- If E_i, i = 1, 2,... are mutually exclusive events then

$$P\left(\bigcup_{i=1}^{n} E_i|F\right) = \sum_{i=1}^{n} P(E_i|F)$$

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Proof?

Example 9

A and B alternate rolling a pair of dice, stopping either when A rolls the sum 9 or when B rolls the sum of 6. Assuming that A rolls first, find the probability that the final roll is made by A?