# Axioms of Probability

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September 10, 2015

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# Sample Space

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- If experiment is flipping of two coins, sample space is {(H,H), (H,T), (T,H), (T,T)}.
- In an experiment, die is rolled continuously until a 6 appears, at which point the experiment stops. What is the sample space of this experiment?

# Example 1

A system is composed of 5 components, each of which is either working (1) or failed (0). Consider a experiment which checks the status of each component and let the outcome be
$(x_1, x_2, x_3, x_4, x_5)$.
What is the sample space?

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What is the sample space?
How many outcomes are in it?

# Event

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- If experiment is flipping of two coins, event that the first coin is head is $\{$(H,H), (H,T)$\}$.
- Two dice is thrown. Event that the sum of dice is odd? Event that at least one of the dice lands on 1?

# Definitions

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- Venn Diagrams? What if there are more than 2 events?

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- What is $E \cup F$?
- What is $E \cap F$?

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Distributive Laws:

$(E \cup F) \cap G = (E \cap G) \cup (F \cap G) \qquad (E \cap F) \cup G = (E \cup G) \cap (F \cup G)$

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# Example 3

Hospital codes incoming patients according to whether they have insurance (code 1 if they do and 0 if they don't) and according to their condition (which is rated as good (g), fair (f) or serious (s)). Experiment is coding a patient.

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- B = patient is uninsured. Outcomes in B?
- $B^c \cup A$?

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Proof for two sets is easy. Proof for n sets?

# What is Probability?

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- How do you know that there would be convergence and that it is unique?

# Three Axioms of Probability

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$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$$

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What is the probability of getting an even number on a die?

# Example 4

An insurance company offers 2 kinds of policies to home owners and to automobile owners. The accompanying table gives proportions for the various categories of policyholders who have both types of insurance. For example, the proportion of individuals with both low homeowner's deductible and low auto deductible is .06

| | Homeowner's | | | |
|---|---|---|---|---|
| Auto | N | L | M | H |
| L | .04 | .06 | .05 | .03 |
| M | .07 | .10 | .20 | .10 |
| H | .02 | .03 | .15 | .15 |

# Example 4 contd

Suppose an individual having both types of policies is randomly selected.

a. What is the probability that the individual has a medium auto and a high homeowner's insurance?

b. What is the probability that the individual has a low auto insurance? A low homeowner's insurance?

c. What is the probability that the individual is in the same category for both auto and homeowner's insurance?

d. Based on your answer in part (c), what is the probability that the two categories are different?

e. What is the probability that the individual has at least one low insurance level?

f. Using the answer in part (e), what is the probability that neither insurance level is low?

# Propositions

## Proposition 1

$P(E^c) = 1 - P(E)$

# Propositions

## Proposition 2

If $E \subset F$ then $P(E) \leq P(F)$

# Propositions

### Proposition 3

$P(E \cup F) = P(E) + P(F) - P(E \cap F)$

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# Propositions

## Proposition 4

$$P\left(E_1 \cup E_2 \cup \ldots \cup E_n\right) = \sum_{i=1}^{n} P(E_i) - \sum_{i_1<i_2}^{n} P(E_{i_1}E_{i_2})$$

$$+ (-1)^{r+1} \sum_{i_1<i_2<\ldots<i_r}^{n} P(E_{i_1}E_{i_2}\ldots E_r) +$$

$$\ldots + (-1)^{n+1} P(E_1E_2\ldots E_r)$$

# Equally Likely Outcomes

$$P(E) = \frac{\text{number of outcomes in E}}{\text{number of outcomes in S}}$$

# Example 6

There are n socks, 3 of which are red, in a drawer. What is the value of n if, when 2 of the socks are chosen randomly, the probability that they are both red is 0.5?

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(c) If 2 students are chosen randomly, what is the prob. that at least one is taking a language class?

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- P(four alike) $= .0193$
- P(five alike) $= .0008$


Probabilities can be used for many purposes. *(Source:* Peanuts® *reprinted by permission of United Feature Syndicate, Inc.)*