Combinatorial Analysis

Chapter 1, Sheldon Ross

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Basic Principle of Counting

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each of the outcomes of first two experiments there are n_3 possible outcomes of the third experiment, ...
then there is a total of $n_1 \cdot n_2 \cdots n_r$ outcomes of r experiments.

Example 1

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- Repetition allowed but last digit of the numbers can not be odd.

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$(0! = 1)$

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- In how may ways can 3 boys (R, L, B) and 3 girls (S, G, N) sit in a row if no two people of the same sex are allowed to sit together?

Ordered Arrangements with Duplicates

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$$\frac{n!}{n_1!n_2!\cdots n_r!}$$

different combinations of n objects are possible.

Example 3

How many different arrangements can be made from the letters

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- Fluke?
- Propose?
- Mississippi?

Example 4

A child has 12 blocks, of which 6 are black, 4 are red, 1 is white and 1 is blue. If the child puts the blocks in a line, how many arrangements are possible?

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- Do we want this?
- Hence divide it by the number of 'repeats' $3 \cdot 2 \cdot 1$.

“n choose r”

We define $\begin{pmatrix} n \\ r \end{pmatrix}$, for $r \leq n$ by

$$\begin{pmatrix} n \\ r \end{pmatrix} = {}^nC_r = \frac{n!}{(n-r)!r!}$$

Example 5

A dance class consists of 22 students, of which 10 are women and rest are men. If 5 men and 5 women are to be chosen and paired off, how many results are possible?

Example 6

A student has to sell 2 books from a collection of 6 math, 7 science and 4 economics books. How many choices are possible if

- both books are to be on the same subject?
- the books are to be of different subjects?

Combinatorial Identity

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

Prove This. Intuition?

Binomial Coefficients

The binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) x^k y^{n-k}$$

Multinomial Coefficients

How to divide a set of distinct n items into r distinct groups of respective size n_1, n_2, $\cdots$, n_r?, where $\sum_i n_i = n$?

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If $n_1 + n_2 + \cdot\cdot + n_r = n$ then

$$\left(\begin{array}{c} n \\ n_1 \cdot n_2 \cdots n_r \end{array} \right) = \frac{n!}{n_1! n_2! \cdots n_r!}$$

represents the number of possible divisions of n distinct objects into r distinct groups of respective size n_1, n_2, $\cdot\cdot$, n_r.

Multinomial Theorem

$$(x_1+x_2+\cdots+x_r)^n = \sum_{(n_1,\ldots,n_r):n_1+\ldots+n_r=n} \binom{n}{n_1 \cdot n_2 \cdots n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$$

Example 7

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Example 8

Suppose that ten people, including you and a friend, line up for a group picture. How many ways can the photographer rearrange the line if she wants to keep exactly three people between you and your friend?

Example 9

A three-digit number is to be formed from the digits 1 through 7, with no digit being used more than once. How many such numbers would be less than 289?

Example 10

In how many ways can the digits 1 through 9 be arranged such that (a) all the even digits precede all the odd digits?

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(b) all the even digits are adjacent to each other?
(c) two even digits begin the sequence and two even digits end the sequence?